SYDNEY TECHNICAL HIGH SCHOOL

Name: ________________

Teacher : ______________

Mathematics Extension 2

HSC ASSESSMENT TASK 2

TERM 2 - 2008

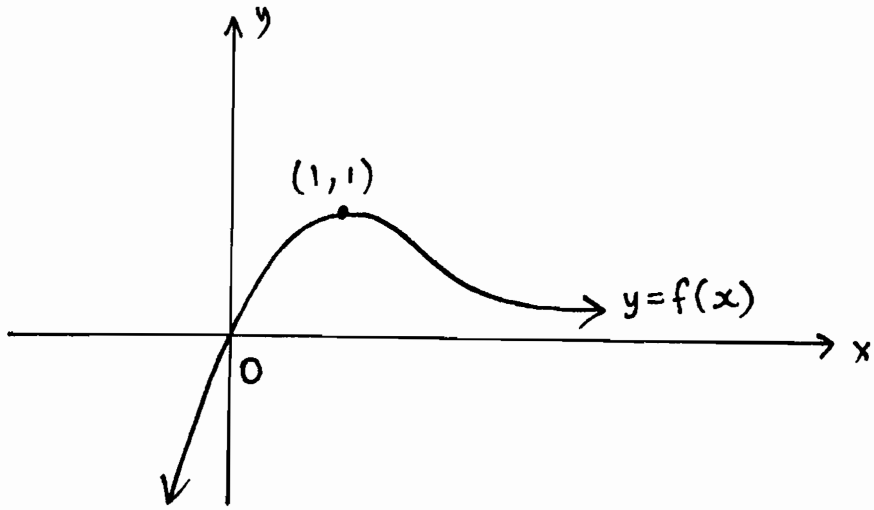
General instructions

- Working time allowed – 70 minutes
- Write using black or blue pen
- Approved calculators may be used
- All necessary working should be shown
- Start each question on a new page
- Attempt all questions

QUESTION 1	QUESTION 2	QUESTION 3	TOTAL

QUESTION 1 (18 Marks) Start a new page

a)



Sketch on separate diagrams the graphs of

i $y = f(x+2)$ **1**

ii $y = f\,|x|$ **2**

iii $y^2 = f(x)$ **2**

iv $y = \ln f\,(x)$ **2**

b) Evaluate

i $\int_1^3 x \ln x \; dx$ **3**

ii $\int_0^{\pi} \sin^3 x \, dx$ **3**

c) Find $\int \frac{dx}{\sqrt{5+4x-x^2}}$ **3**

d) Evaluate $\int_0^1 \frac{2}{\sqrt{x^2+1}} \; dx$, leaving your answer in exact form **2**

QUESTION 2 (16 Marks) Start a new page

a) i Find the real numbers A, B and C such that

$$\frac{x-2}{(x^2+4)(x+1)} = \frac{Ax+B}{x^2+4} + \frac{C}{x+1}$$ **3**

ii Hence, find $\int \frac{x-2}{(x^2+4)(x+1)}\,dx$ **3**

b) Find the integers m and n such that $(x+1)^2$ is a factor of $x^5 + 2x^2 + mx + n$ **2**

c) If α, β, δ are the roots of the polynomial $4x^3 + 8x^2 - 1 = 0$

i Find a polynomial with roots α^2, β^2 and δ^2 **2**

ii Find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\delta^2}$ **2**

d) Use the substitution $x = 2\sin\theta$ to find $\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$ **4**

QUESTION 3 (16 Marks) Start a new page

a) If $ax^4 + bx^3 + dx + e = 0$ has a triple root, show that $4a^2d + b^3 = 0$ given that $a, b, d,$ and e are non zero integers. **3**

b) $P\left(cp, \frac{c}{p}\right)$ and $Q(cq, \frac{c}{q})$ are two variable points on the rectangular hyperbola $xy = c^2$, so that the points P, Q and $S\left(c\sqrt{2},\ c\sqrt{2}\right)$, are always collinear. **8**

The tangents to the hyperbola at P and Q intersect at R.

i Show that the tangent to the hyperbola $xy = c^2$ at the point $T\left(ct, \frac{c}{t}\right)$ has equation $x + t^2y = 2ct$

ii Show that R has co-ordinates $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$

iii Show that $p + q = \sqrt{2}\,(1 + pq)$

iv Hence, find the equation of the locus of R

c) i Find the complex solutions of $z^7 = 1$ **3**

ii Hence, by factorizing over the real numbers, prove that

$$\cos\frac{3\pi}{7} + \cos\frac{\pi}{7} - \cos\frac{2\pi}{7} = \frac{1}{2}$$ **2**

END OF TEST

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \quad \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

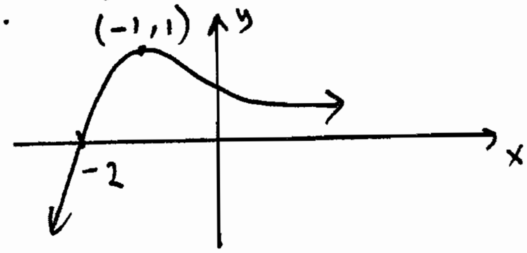
EXTENSION 2 : TASK II (2008)

SOLUTIONS

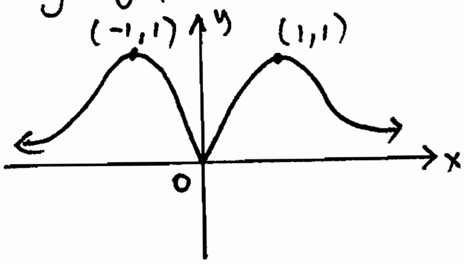
Question 1

a) $y = f(x+2)$

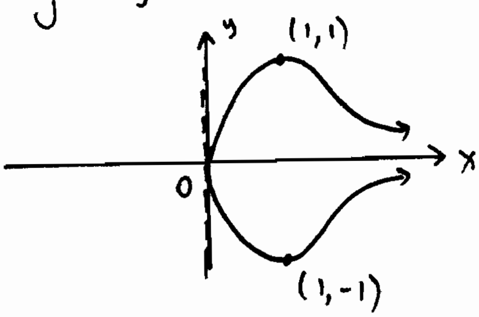
i.



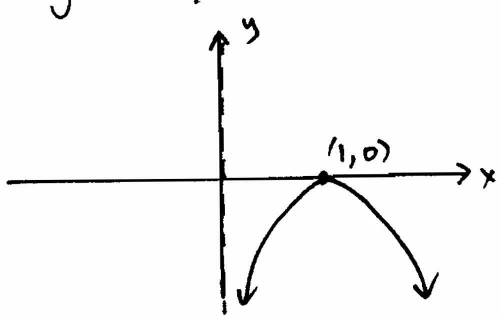
ii. $y = f|x|$



iii. $y^2 = f(x)$



iv. $y = \ln f(x)$



b) i. $\int_1^3 x \ln x \, dx$

$= \ln x \cdot \frac{1}{2}x^2 - \int \frac{1}{2}x^2 \cdot \frac{1}{x} \, dx$

$= \frac{x^2 \ln x}{2} - \frac{1}{2}\int x \, dx$

$= \frac{x^2 \ln x}{} - \frac{x^2}{} \Big]^3$

ii. $\int_0^\pi \sin x (1 - \cos^2 x) \, dx$

$= \int_0^\pi \sin x - \sin x (\cos x)^2 \, dx$

$= \left[-\cos x + \frac{1}{3}\cos^3 x\right]_0^\pi$

$= -\cos\pi + \frac{1}{3}(\cos\pi)^3 - \left[-\cos 0 + \frac{1}{3}(\cos 0)^3\right]$

$= 1 + \frac{1}{3} \times -1 - \left[-1 + \frac{1}{3}\right]$

$= 4/3$

c) $\int \frac{dx}{\sqrt{5 + 4x - x^2}}$

$= \int \frac{dx}{\sqrt{9 - (x-2)^2}}$

$= \sin^{-1}\left(\frac{x-2}{3}\right) + C$

d) $\int_0^1 \frac{2}{\sqrt{x^2+1}} \, dx$ Standard $\int$ sheet

$= 2\ln\left[x + \sqrt{x^2+1}\right]_0^1$

$= 2\left[\ln(1+\sqrt{2}) - 0\right]$

$= 2\ln(1+\sqrt{2})$

Question 2

a)

$x-2=(Ax+B)(x+1)+C(x^2+4)$

let $x=-1$

$-3=5C$

$C=-3/5$

let $x=0$

$B=2/5$

let $x=1$

$-1=(A+2/5)(2)+-3/5(5)$

$A=3/5$

ii. $\int \frac{3x+2}{5(x^2+4)} - \frac{3}{5(x+1)}\, dx$

$=\int \frac{3x}{5(x^2+4)} + \frac{2}{5(x^2+4)} - \frac{3}{5(x+1)}\, dx$

$=\frac{3}{10}\ln(x^2+4) + \frac{1}{5}\tan^{-1}\frac{x}{2} - \frac{3}{5}\ln(x+1) + C$

b) $P(x)=x^5+2x^2+mx+n$

$P'(x)=5x^4+4x+m$

$P'(-1)=0$

$5-4+m=0$

$m=-1$

$P(-1)=0$

$-1+2-m+n=0$

$n=-2$

c) $4x^3+8x^2-1=0$

let $x=\sqrt{x}$

$4\sqrt{x}^3+8\sqrt{x}^2-1=0$

$4x\sqrt{x}+8x-1=0$

$4x\sqrt{x}=1-8x$

$16x^2 \cdot x = 1-16x+64x^2$

$16x^3-64x^2+16x-1=0$

ii. $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}$

let $x=1/x$

$16\left(\frac{1}{x}\right)^3-64\left(\frac{1}{x}\right)^2+16\left(\frac{1}{x}\right)-1=0$

$16-64x+16x^2-x^3=0$

$x^3-16x^2+64x-16=0$

$\therefore \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=\frac{-b}{a}$

$=16$

d) $\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\, dx$

$x=2\sin\theta$

$\frac{dx}{d\theta}=2\cos\theta$

$dx=2\cos\theta\, d\theta$

$x=1 \quad \theta=\pi/6$

$x=0 \quad \theta=0$

$=\int_0^{\pi/6} \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}} \cdot 2\cos\theta\, d\theta$

$=\int_0^{\pi/6} \frac{8\sin^2\theta\cos\theta\, d\theta}{2\cos\theta}$

$=4\int_0^{\pi/6} \sin^2\theta\, d\theta$

$2\int 1-\cos 2\theta\, d\theta$

$=2\left[\theta-\frac{1}{2}\sin 2\theta\right]_0^{\pi/6}$

$=2\left[\pi/6-\frac{\sqrt{3}}{4}-(0)\right]$

$=\frac{\pi}{3}-\frac{\sqrt{3}}{2}$

Question 3.

a) $P(x) = ax^4 + bx^3 + dx + e$
$P'(x) = 4ax^3 + 3bx^2 + d$
$P''(x) = 12ax^2 + 6bx = 0$

$x(12ax + 6b) = 0$

$x = 0$ is not Triple root as $P(0) \neq 0$
$\therefore 12ax + 6b = 0$

$x = -\frac{6b}{12a}$

$x = -\frac{b}{2a}$

$\therefore P'\left(\frac{-b}{2a}\right)\ 4a\left(\frac{-b}{2a}\right)^3 + 3b\left(\frac{-b}{2a}\right)^2 + d = 0$

$-\frac{4ab^3}{8a^3} + \frac{3b^3}{4a^2} + d = 0$

$\frac{-2b^3}{4a^2} + \frac{3b^3}{4a^2} + d = 0$

$\frac{b^3}{4a^2} + d = 0$

ie $4a^2d + b^3 = 0$

b) $xy = c^2$
$y = \frac{c^2}{x}$ $\quad y' = -\frac{c^2}{x^2}$ at $x = cp$

$m_T = \frac{-c^2}{c^2p^2}$
$= -1/p^2 \quad (-1/t^2)$

$\therefore y - c/t = -1/t^2 (x - ct)$

$t^2y - tc = -x + ct$
$x + t^2y = 2ct$

at P $\quad x + p^2y = 2cp$

at Q $\quad x + q^2y = 2cq$

Finding R $\quad (p-q)(p+q)y = 2c(p-q)$

$x = 2cp - p^2\left(\frac{2c}{p+q}\right) = \frac{2cpq}{p+q}$

$R = \left[\frac{2cpq}{p+q}, \frac{2c}{p+q}\right]$

III. Gradient PS = Gradient QS

$\frac{c/p - c\sqrt{2}}{cp - c\sqrt{2}} = \frac{c/q - c\sqrt{2}}{cq - c\sqrt{2}}$

$\frac{1/p - \sqrt{2}}{p - \sqrt{2}} = \frac{1/q - \sqrt{2}}{q - \sqrt{2}}$

$\frac{q}{p} - \frac{\sqrt{2}}{p} - \sqrt{2}q + 2 = \frac{p}{q} - \sqrt{2}p - \sqrt{2}/q + 2$

$q^2 - q\sqrt{2} - \sqrt{2}pq^2 = p^2 - p\sqrt{2} - \sqrt{2}p^2q$

$\sqrt{2}p(1+pq) - \sqrt{2}q(1+pq) = (p-q)(p+q)$

$\sqrt{2}(p-q)(1+pq) = (p-q)(p+q)$

$\therefore p + q = \sqrt{2}(1+pq)$

IV. $\frac{p+q}{\sqrt{2}} - 1 = pq$

$x = \frac{2cpq}{p+q}$, $\quad y = \frac{2c}{p+q}$

$x(p+q) = 2cpq$, $\quad (p+q) = \frac{2c}{y}$

$x\left[\frac{2c}{y}\right] = 2c\left[\frac{p+q}{\sqrt{2}} - 1\right]$

$\frac{2xc}{y} = 2c\left[\frac{2c/y}{\sqrt{2}} - 1\right]$

$xc = cy\left[\frac{2c}{\sqrt{2}y} - 1\right]$

$x = y\left[\frac{\sqrt{2}c}{y} - 1\right]$

d) $z^7 = 1$

$z = 1$ is a sol

$z = 1 \operatorname{cis} 0$

$z = \left[1 \operatorname{cis}(0 + 2\pi k)\right]^{1/7}$

$z = \operatorname{cis}\left(\frac{0 + 2\pi k}{7}\right)$

$k = 0 \quad z = 1$

$k = 1 \quad z = \operatorname{cis} 2\pi/7$

$k = 2 \quad z = \operatorname{cis} 4\pi/7$

$k = 3 \quad z = \operatorname{cis} 6\pi/7$

$k = 4 \quad z = \operatorname{cis} 8\pi/7$

$k = 5 \quad z = \operatorname{cis} 10\pi/7$

$k = 6 \quad z = \operatorname{cis} 12\pi/7$

ii) sum roots

$$\operatorname{cis}\frac{2\pi}{7} + \operatorname{cis}\frac{4\pi}{7} + \operatorname{cis}\frac{6\pi}{7} + \operatorname{cis}\frac{8\pi}{7} + \operatorname{cis}\frac{10\pi}{7} + \operatorname{cis}\frac{12\pi}{7} = -1$$

$$2\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{8\pi}{7} = -1$$

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} = -\frac{1}{2}$$

$$\cos\frac{2\pi}{7} - \cos\frac{3\pi}{7} - \cos\frac{\pi}{7} = -\frac{1}{2}$$

$$-\cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{\pi}{7} = -\frac{1}{2}$$

$$\cos\frac{3\pi}{7} + \cos\frac{\pi}{7} - \cos\frac{2\pi}{7} = \frac{1}{2}.$$